

# DECAY OF CORRELATIONS FOR MAXIMAL MEASURE OF MAPS DERIVED FROM ANOSOV: I: MOSTLY CONTRACTING CENTER

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ABSTRACT. It was proven by Ures that  $C^1$  diffeomorphism of  $\mathbb{T}^3$  that is derived from Anosov admits a unique maximal measure. Here we show that the maximal measure has exponential decay of correlations for Hölder observables, assuming the middle eigenvalue of the linear Anosov model is contracting.

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## 1. INTRODUCTION

By the early 1970s, Brin, Pesin [8] and Pugh, Shub [31] began the study of partially hyperbolic diffeomorphisms, as an extension of the classical class of Anosov diffeomorphisms.

A diffeomorphism  $f$  on a compact manifold  $M$  is *partially hyperbolic* if there is a  $Df$ -invariant splitting of the tangent bundle  $TM = E^s \oplus E^c \oplus E^u$ , such that all unit vectors  $v^i \in E_x^i \setminus \{0\}$  ( $i = s, c, u$ ) with  $x \in M$  for some suitable Riemannian metric satisfies

$$e^{\lambda_1(x)} \leq |df|_{E_x^s}(v^s) \leq e^{\lambda_2(x)},$$

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$$\begin{aligned} e^{\lambda_3(x)} &\leq |df|_{E_x^c}(v^c)| \leq e^{\lambda_4(x)}, \\ e^{\lambda_5(x)} &\leq |df|_{E_x^u}(v^u)| \leq e^{\lambda_6(x)}, \end{aligned}$$

where  $\lambda_1(x) \leq \lambda_2(x) < \lambda_3(x) \leq \lambda_4(x) < \lambda_5(x) \leq \lambda_6(x)$  and  $\lambda_2(x) < 0$ ,  $\lambda_5(x) > 0$ .

We are interested in three dimensional *derived Anosov diffeomorphisms*,  $\mathcal{D}(A)$ , which are the partially hyperbolic diffeomorphisms in the same isotopy class with some linear Anosov diffeomorphism  $A$ . This definition is a generalization of the classical construction of partially hyperbolic, robustly transitive diffeomorphisms by Mañé [28]. Let us mention that, although in the same isotopy class, the dynamics of a derived Anosov diffeomorphism and the linear Anosov one can be quite different. For example, the center exponent of a volume preserving derived Anosov diffeomorphism may have different sign with the center exponent of the linear Anosov diffeomorphism ([29]).

But from another direction, in the last decade, people began to realize that, the derived Anosov diffeomorphism does inherit topological hyperbolicity from its isotopy class (see for example, [7, 18, 30, 19]). This weak hyperbolicity was used in [20, 34, 35] to deduce measure-theoretical properties for derived Anosov diffeomorphisms.

By the variation principle, for any invariant probability measure of a diffeomorphism, the metric entropy is always bounded by the topological entropy. An invariant probability is called *maximal* if the corresponding metric entropy coincides with the topological entropy of this diffeomorphism. In another word, the maximal measures are the measures which are most complicated. It is a well-known fact that every transitive Anosov diffeomorphism admits a unique maximal measure, and this maximal measure has exponential decay of correlations for Hölder continuous observables (see for instance [6]). It was observed by Ures in [34] that every  $C^1$  diffeomorphism  $f \in \mathcal{D}(A)$  also admits a unique maximal measure. Denote this maximal measure of  $f$  by  $\nu_f$ , in this paper we are going to prove the following:

**Theorem A.**<sup>1</sup> *Suppose  $A$  is a three dimensional linear Anosov diffeomorphism over  $\mathbb{T}^3$  with center exponent negative. Then for any  $C^1$  diffeomorphism  $f \in \mathcal{D}(A)$ , its maximal measure  $\nu_f$  has exponential decay of correlations for Hölder continuous observables: for  $0 < \gamma < 1$  there exists some constants  $0 < \tau < 1$  such that for all  $\phi, \psi \in C^\gamma(M)$  there exists  $K(\phi, \psi) > 0$  satisfying*

$$\left| \int (\phi \circ f^n) \psi d\nu_f - \int \phi d\nu_f \int \psi d\nu_f \right| \leq K(\phi, \psi) \tau^n, \text{ for every } n \geq 1.$$

We also obtain large deviation estimate for  $C^0$  functions:

**Theorem B.** *For every  $\phi \in C^0(M)$  with  $\nu_f(\phi) = 0$  and every  $\epsilon > 0$  there exists constants  $C_\epsilon, c_\epsilon > 0$  such that*

$$\nu_f(|S_n(\phi)| > \epsilon n) \leq C_\epsilon e^{-c_\epsilon n}.$$

The proof of this theorem can be found in Section 5.

## 2. PRELIMINARY

Throughout this paper, we assume  $A : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  to be a linear hyperbolic torus automorphism with eigenvalues  $0 < \kappa_1 < \kappa_2 < 1 < \kappa_3$  and eigenspaces  $E_1, E_2, E_3$  respectively, and  $f \in \mathcal{D}(A)$  to be a derived Anosov diffeomorphism.

<sup>1</sup> In [10], a different kind of derived Anosov diffeomorphism was considered, where they assume the existence of Markov partition, some 'good' component where the center direction is uniformly expanding, and on the 'bad' components the center direction does not contract too much, that is, the small norm is bounded from below by a value close to one (condition (5)). With  $C^{1+}$  regularity assumption, they proved similar results. We thank Paulo Varandas for the discussion on this work.

We treat  $A$  as a partially hyperbolic diffeomorphism with invariant subbundles  $E_A^s = E_1$ ,  $E_A^c = E_2$  and  $E_A^u = E_3$ . Denote by  $\omega$  the maximal measure of  $A$ , it is well-known that,  $\omega$  is indeed the volume measure. We also denote by  $\mathcal{F}_A^i$  ( $i = s, c, u, cs, cu$ ) the linear foliation tangent to the subbundles  $E_A^s, E_A^c, E_A^u, E_A^{cs} = E_A^s \oplus E_A^c, E_A^{cu} = E_A^c \oplus E_A^u$  respectively.

**2.1. Dynamical coherence.** By Franks [16], there exists a continuous surjective map  $h : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  which semiconjugates  $f$  to  $A$ :  $h \circ f = A \circ h$ . The following proposition shows that every three dimensional derived Anosov diffeomorphism admits a weak form of topological hyperbolicity.

**Proposition 2.1.**  *$f$  is dynamically coherent, the Franks' semi-conjugation  $h$  maps the center stable, center, center unstable and unstable leaves of  $f$  into the corresponding leaves of  $A$ . Moreover,*

- (a) *restrict to each unstable leaf of  $f$ ,  $h$  is bijective;*
- (b) *there is  $K > 0$  depending only on  $f$ , such that for every  $x \in \mathbb{T}^3$ ,  $h^{-1}(x)$  is either a point, or a connected center segment of  $f$  with length bounded by  $K$ .*
- (c) *the stable, center, unstable foliation of  $f$  are quasi-isometry, that is, there exist  $a, b > 0$  such that for any two points  $\tilde{x}, \tilde{y}$  belonging to the same lifted leaf  $\tilde{\mathcal{F}}^i$  ( $i = s, c, u$ ) in the universal covering space  $\mathbb{R}^3$ ,*

$$d_{\tilde{\mathcal{F}}^i}(\tilde{x}, \tilde{y}) < ad(\tilde{x}, \tilde{y}) + b$$

*where  $d(\cdot, \cdot)$  is the Euclidean metric on  $\mathbb{R}^3$ .*

*Proof.* It is proven by Potrie [30, Theorem A.1] that  $f$  is dynamically coherent. Moreover, in [30, Theorem 7.10] he shows that the semi-conjugation  $h$  maps each center stable leaf of  $f$  to a center stable leaf of  $A$ . By considering the inverse of  $f$ , one may show that  $h$  also maps the center unstable leaf of  $f$  into a center unstable leaf of  $A$ . Because every center leaf of  $f$  belongs to the intersection of the corresponding center stable leaf and center unstable leaf, by the previous discussion,  $h$  maps every center leaf of  $f$  to a center leaf of  $A$ .

The item (a) is proven in [30, Corollary 7.7, Remark 7.8], and item (b) is proven by [34] (see also [35, Proposition 3.1]).  $\mathcal{F}^c$  being quasi-isometry is proven by Hammerlindl and Potrie in [19, Section 3].  $\square$

*Remark 2.2.* By Ledrappier-Walter's formula [25],  $h_*$  preserves metric entropy. In particular,  $h_*(\nu_f) = \omega$

The following proposition shows that, this topological hyperbolicity implies that the ergodic measures with higher entropy for any derived Anosov diffeomorphism  $f$  and for the linear Anosov diffeomorphism  $A$  are essentially the same.

**Proposition 2.3.** [35, Theorem 3.6] *Let  $\mu$  be an ergodic probability measure of  $f$  with  $h_\mu(f) > -\log \kappa_1$ . Then for  $\mu$  almost every  $x$ ,  $h^{-1} \circ h(x) = \{x\}$ , that is, the semi-conjugation  $h$  preserves the ergodic measures with entropy larger than  $-\log \kappa_1$ .*

For the further discussion, we also need the following property:

**Lemma 2.4.** *For every  $x$  belonging to a full measure subset  $\Gamma_A$  of  $\omega$ ,  $h^{-1}(\mathcal{F}_A^u(x))$  consists of a single unstable leaf.*

*Proof.* By Proposition 2.3, there is a  $\nu_f$  full measure subset  $\Gamma$  such that for every  $y \in \Gamma$ ,  $h^{-1} \circ h(y) = y$ . Let  $\Gamma_A = h(\Gamma)$ , then by Remark 2.2,  $\omega(\Gamma_A) = 1$  (for the measurability of  $\Gamma_A$  see [35, Corollary 3.4]).

Then for every point  $x \in \Gamma_A$ ,  $h^{-1}(x)$  consists of a unique point. By Proposition 2.1,  $h^{-1}(\mathcal{F}_A^u(x))$  is a union of unstable leaves of  $f$ . Suppose that  $h^{-1}(\mathcal{F}_A^u(x))$

contains at least two unstable leaf of  $f$ , then by (a) of Proposition 2.1, each unstable leaf contains a pre-image of  $x$ , which contradicts to the fact that  $x \in \Gamma_A$ .  $\square$

**2.2. Markov partition along  $\mathcal{F}^u$ .** In this subsection, we will build a Markov partition along the unstable foliation of  $f$ , and consider the disintegration of  $\nu$  along this partition.

We start with a Markov partition  $\mathcal{M}^A = \{M_1^A, \dots, M_k^A\}$  for the linear Anosov map  $A$ , which enables us to define a partition  $\xi^A$  along the unstable foliation, such that the elements are the connected components of the intersection of each unstable leaf with  $M_i^A$  ( $i = 1, \dots, k$ ). This partition is clearly a Markov partition.

The two partitions  $\mathcal{M}^A$  and  $\xi^A$  above induce similar partitions for  $f$ :

- $\mathcal{M} = \{h^{-1}(M_i^A); i = 1, \dots, k\};$
- $\xi^u = \{\xi^u(x) = (h|_{\mathcal{F}^u(x)})^{-1}(\xi^A(h(x)))\}.$

By the Franks' semi-conjugation and Proposition 2.1 (a),  $\xi^u$  is a Markov partition of  $f$  along the unstable foliation given by the intersection of elements of  $\mathcal{M}$  and  $\mathcal{F}^u$ .

By Proposition 2.3, there is a  $\nu_f$  full measure subset  $\Gamma$  and a  $\omega$  full measure subset  $\Gamma_A$  such that  $h|_{\Gamma}$  is bijective and preserves the measures, that is, for any measurable subset  $X \subset \Gamma$ ,  $\nu(X) = \omega(h(X))$ . By Lemma 2.4,  $h_*^{-1}$  maps the conditional measure  $\omega_{(\cdot)}^u$  of  $\omega$  corresponding to the partition  $\xi^A$  to the conditional measure  $\nu_{(\cdot)}^u$  of  $\nu$  corresponding to the partition  $\xi^u$ . That is, for any  $x \in \Gamma_A$ ,  $(h^{-1})_*\omega_x^u = \nu_{h^{-1}(x)}^u$ .

**2.3. Local product structure.** It is well know that the maximal measure  $\omega$  of the linear Anosov diffeomorphism admits a product structure (for instance, see Bowen [6]). More precisely, for  $i = 1, \dots, k$  and  $x_i^A \in M_i^A$ , denote by  $\mathcal{F}_{A,loc}^*(x_i^A)$  ( $* = cs, u$ ) the connected component of  $\mathcal{F}^* \cap M_i^A$  which contains  $x_i^A$ , then

**Proposition 2.5.** *For each  $i = 1, \dots, k$ , there are measure  $\omega_i^{cs}$  and  $\omega_i^u$  supported on  $\mathcal{F}_{A,loc}^{cs}(x_i)$  and  $\mathcal{F}_{A,loc}^u(x_i)$  respectively, such that*

$$\omega|_{M_i^A} = \omega_i^{cs} \times \omega_i^u.$$

The above proposition implies that in each foliation chart  $M_i^A$ , the center-stable holonomy preserves the conditional measures  $\omega^u(\cdot)$ .

For every  $1 \leq i \leq k$ , fix any  $x_i \in h^{-1}(x_i^A)$ . Recall that by Proposition 2.1,  $h$  preserve the unstable and center-stable foliations, and restrict to each unstable leaf of  $f$ ,  $h$  is bijective. Because  $h_*(\nu_f) = \omega$ , we have that:

**Proposition 2.6.** *For every  $i = 1, \dots, k$ , there are measures  $\nu_i^{cs}$  and  $\nu_i^u$  supported on  $\mathcal{F}_{loc}^{cs}(x_i)$  and  $\mathcal{F}_{loc}^u(x_i)$  such that*

$$\nu|_{M_i} = \nu_i^{cs} \times \nu_i^u.$$

*This implies that for any  $y \in \Gamma \cap M_i$ , the conditional measure  $\nu_y^u = (H_{x_i,y}^{cs})_*(\nu_i^u)$ , where  $H_{x_i,y}^{cs} : \xi^u(x_i) \rightarrow \xi^u(y)$  denotes the center stable holonomy map induced by  $\mathcal{F}^{cs}$ . In particular, for any  $y, z \in \Gamma \cap M_i$ ,  $H_{y,z}^{cs}$  maps the conditional measure  $\nu_y^u$  to  $\nu_z^u$  with Jacobian equal to 1.*

By the above proposition, we can indeed extend the family of conditional measures  $\{\nu_x^u : x \text{ in some full measure subset } \Gamma\}$  to the whole  $\mathbb{T}^3$ . More precisely, for every  $x \in M_i$ , define

$$\nu_x^u = (H_{x_i,x}^{cs})_*\nu_i^u.$$

**Remark 2.7.** For every point  $x \in h^{-1}(\partial M_i^A)$  ( $i = 1, \dots, k$ ), there are many different elements  $\xi^u(x)$  as above, and also different measures  $\nu_x^u$ . In the following we treat these elements as disjoint, and consider all the possible measures. More precisely, we will think the closure of  $h^{-1}(M_i^A)$  ( $i = 1, \dots, k$ ),  $\text{Cl}(h^{-1}(M_i^A))$ , are all disjoint, and consider the disintegration of  $\nu_f$  inside each component.

By the uniqueness of disintegration, we have:

**Lemma 2.8.** *For every  $x \in M$ ,*

$$\frac{df_*(\nu_x^u) |_{\xi^u(f(x))}}{d\nu_{f(x)}^u}(f(x)) = \nu_x^u(f^{-1}\xi^u(x)).$$

**2.4. High iteration.** In this subsection we claim that, to prove Theorem A for  $f$ , it suffices to consider its iteration  $f^N$  for any  $N > 0$ .

This is because  $f^N$  belongs to the isotopy class of  $A^N$ , we may apply [34] to show that  $\nu_f$  is still the unique maximal measure of  $f^N$ . Suppose Theorem A holds for  $f^N$  with  $N > 0$ , that is, for any  $\phi, \psi \in C^\gamma(M)$  there exists  $K_N(\phi, \psi) > 0$  satisfying

$$| \int (\phi \circ f^{nN}) \psi d\nu_f - \int \phi d\nu_f \int \psi d\nu_f | \leq K_N(\phi, \psi) \tau^n = (\tau^{1/N})^{nN}, \text{ for every } n \geq 1.$$

Then for  $m = nN + i$  with  $1 \leq i < N$ ,

$$\begin{aligned} & | \int (\phi \circ f^m) \psi d\nu_f - \int \phi d\nu_f \int \psi d\nu_f | \\ &= | \int (\phi \circ f^i) \circ f^{nN} \psi d\nu_f - \int \phi \circ f^i d\nu_f \int \psi d\nu_f | \\ &\leq K_N(\phi \circ f^i, \psi) (\tau^{1/N})^{nN} \\ &= \frac{K_N(\phi \circ f^i, \psi)}{\tau^{i/N}} (\tau^{1/N})^m. \end{aligned}$$

Take  $K(\phi, \psi) = \max_{i=0, \dots, N-1} \{ \frac{K_N(\phi \circ f^i, \psi)}{\tau^{i/N}} \}$ , we conclude the proof of Theorem A.

### 3. NEGATIVE CENTER EXPONENT

In this section we show that the center Lyapunov exponent of  $\nu_f$  is negative, which was proven in [34] for every  $C^{1+\alpha}$  derived Anosov diffeomorphism.<sup>2</sup> For completeness, we cite the proof here, and show that this proposition indeed holds for any  $C^1$  derived Anosov diffeomorphism.<sup>3</sup>

**Theorem 3.1.** *Suppose  $f$  is a  $C^1$  diffeomorphism belongs to  $\mathcal{D}(A)$ , then  $\lambda^c(\nu_f) \leq \log \kappa_2 < 0$ .*

*Proof.* First, let us observe that three dimensional partially hyperbolic diffeomorphisms are always  $C^1$  away from tangencies (see [36]). Hence, by [27], the metric entropy varies upper-semi continuously with respect to the invariant measures and the diffeomorphisms.

By Remark 2.2, for any diffeomorphism  $g \in \mathcal{D}(A)$ , the topological entropy is always equal to  $h_{top}(A)$ . Moreover, by [34],  $g$  admits a unique maximal measure  $\nu_g$ . If we take a sequence of  $C^2$  diffeomorphisms  $g_n \rightarrow f$  in the  $C^1$  topology and  $\mu$  an accumulation point of  $\nu_{g_n}$ , we get that

$$h_\mu(f) \geq \limsup_n h_{\nu_{g_n}}(g_n) = h_{top}(A).$$

<sup>2</sup>In [34], the partially hyperbolic diffeomorphism is supposed to be uniform absolute, that is, the parameters  $\lambda_i(x)$ ,  $i = 1, \dots, 6$  in the definition of partially hyperbolic diffeomorphism do not depend on  $x$ .

<sup>3</sup>Instead of citing the Pesin-Ruelle-like inequality of [24] in the original proof, we make use of Ledrappier-Young's entropy formula and [11].

By the uniqueness of the maximal measure of  $f$ , we have  $\mu = \nu_f$ . Hence, we conclude that the maximal measure  $\nu_f$  varies continuously respect to the diffeomorphisms. Because the center bundle is one dimensional, we get that

$$\lambda^c(\nu_f) = \int \log |df|_{E^c}(x) d\nu_f(x)$$

varies continuously respect to the diffeomorphisms. Therefore, it suffices for us to prove the above theorem for  $C^2$  diffeomorphisms.

Applying the Ledrappier-Young's dimension formula for  $f^{-1}$  ([26, Proposition 7.3.2 and Theorem  $C'$ ]), we have that

$$(1) \quad h_{\nu_f}(f) \leq -\lambda^c(\nu_f) + h_{\nu_f}(\mathcal{F}^s, f^{-1}),$$

where  $h_{\nu_f}(\mathcal{F}^s, f^{-1})$  is the partial entropy along the stable foliation. We need the following definition:

*Definition 3.2.* Suppose  $\mathcal{F}$  is an invariant foliation of a diffeomorphism  $f$ , the *geometric expanding speed* of  $f$  along  $\mathcal{F}$  is defined as

$$\chi(\mathcal{F}, f) = \sup_{D \text{ is a leaf compact set}} \limsup \frac{1}{n} \log \text{vol}^{\mathcal{F}}(f^n(D)).$$

By [11], this partial entropy is bounded by the geometric expanding speed:

$$h_{\nu_f}(\mathcal{F}^s, f^{-1}) \leq \chi(\mathcal{F}^s, f^{-1}).$$

*Remark 3.3.* The original statement of [11] is for the Pesin unstable lamination, but the argument also holds for Pesin strong unstable lamination.

Now we are going to show that

$$\chi(\mathcal{F}^s, f^{-1}) \leq -\log \kappa_1.$$

Because  $\mathcal{F}^s$  is quasi-isometry in the universal covering space ((c) of Proposition 2.1), we only need to show that for any  $x, y$  contained in the same  $\mathcal{F}^s$  leaf,

$$\limsup \frac{1}{n} \log(d(\tilde{f}^{-n}(\tilde{x}), \tilde{f}^{-n}(\tilde{y}))) \leq -\log \kappa_1,$$

where  $\tilde{f}$  is any lift of  $f$ , and  $\tilde{x}, \tilde{y}$  are lifts of  $x, y$  in  $\mathbb{R}^3$  such that they locate in the same leaf of a lift of  $\mathcal{F}^s(x)$ . Since the Franks' semi-conjugacy is in the same isotopy class of the identity map, every lift  $\tilde{h}$  of  $h$  in the universal covering space  $\mathbb{R}^3$  satisfies that  $d(\tilde{h}(\tilde{x}), \tilde{x})$  is uniformly bounded from above. Thus, we only need to show that

$$\limsup \frac{1}{n} \log(d(A^{-n}(\tilde{h}(\tilde{x})), A^{-n}(\tilde{h}(\tilde{y})))) \leq -\log \kappa_1.$$

But this is clearly true, we conclude the proof of the claim.

Let us continue to the prove of Theorem 3.1. By (1), and the fact that  $\nu_f$  is a maximal measure, we have

$$-\lambda^c(\nu_f) + h_{\nu_f}(\mathcal{F}^s, f^{-1}) \geq h_{\nu_f}(f) = -\log \kappa_1 - \log \kappa_2.$$

By the above discussion,  $-\lambda^c(\nu_f) - \log \kappa_1 \geq -\log \kappa_1 - \log \kappa_2$ . Hence,  $\lambda^c(\nu_f) \leq \log \kappa_2 < 0$ . We conclude the proof.  $\square$

4. CLASSIFICATION OF  $\nu_f$ 

In this section we are going to build contracting property along  $E^{cs}$  bundle for every element of  $\xi^u$  (Proposition 4.3). The argument depends on the following two properties:

- the maximal measure is unique;
- the conditional measure  $\nu_x^u$  of  $\nu_f$  along the partition  $\xi^u$  is invariant under center-stable holonomy map (Proposition 2.6).

As we have already mentioned, these two properties are consequences of the topological hyperbolicity (Proposition 2.1) of the derived Anosov diffeomorphism.

**4.1. Special probability measure spaces.** In this subsection we are going to introduce a special class of probability measures which are defined on unstable plaques. A similar definition was used by Dolgopyat in [12, Section 5] (with different reference measure) to study physical measures.

Fix  $0 < \gamma < 1$  which denotes the regularity of Hölder functions. Let  $E_1(R)$  the set of measures  $l$  such that:

$$l(A) = \int_{\xi^u(x)} A(z) e^{G(z)} d\nu_x^u(z),$$

where  $l(1) = 1$  and  $|G(z_1) - G(z_2)| \leq R d^\gamma(z_1, z_2)$  for any  $z_1, z_2 \in \xi^u(x)$ .

Let  $E_2(R)$  be the convex hull of  $E_1(R)$  and  $E(R)$  the closure of  $E_2(R)$  under weak\* topology. The family  $E(R)$  is continuous:  $E(R_0) = \bigcap_{R > R_0} E(R)$  (This follows from the fact that  $E_1(R_0) = \bigcap_{R > R_0} E_1(R)$ ). Denote by  $\mathcal{T}(l) = l(A \circ f)$  the transfer operator, and  $\lambda_5 = \max_x \lambda_5(x)$ .

**Proposition 4.1.**  $\mathcal{T} : E(R) \rightarrow E(R e^{-\lambda_5 \gamma})$ .

*Proof.* It suffices to prove that  $\mathcal{T} : E_1(R) \rightarrow E_2(R e^{-\lambda_5 \gamma})$ .

Take  $l \in E_1(R)$  such that  $\text{supp}(l) \subset \xi^u(x)$  for some point  $x \in M$ . Then

$$\mathcal{T}(l)(A) = \int_{\xi^u(x)} e^{G(x)} A(f(z)) d\nu_x^u(z) = \int_{f(\xi^u(x))} e^{G \circ f^{-1}(y)} A(y) df_* \nu_x^u(y).$$

Let  $f(\xi^u(x)) = \bigcup_i \xi^u(x_i)$ , by the uniqueness of disintegration,  $f_*(\nu_x^u) = \sum c_i \nu_{x_i}^u$ , where  $c_i = \nu_x^u(f^{-1}(\xi^u(x_i)))$ . Then  $\mathcal{T}(l) = \sum c_i l_i$  where

$$l_i(A) = \int_{\xi^u(x_i)} e^{G \circ f^{-1}(y)} A(y) d\nu_{x_i}^u(y).$$

Because  $|(G \circ f^{-1})(y_1) - (G \circ f^{-1})(y_2)| \leq R e^{-\lambda_5 \gamma} d^\gamma(y_1, y_2)$ , we have  $\mathcal{T} : E_1(R) \rightarrow E_2(R e^{-\lambda_5 \gamma})$ .  $\square$

**Lemma 4.2.**  $E(0)$  contains a unique invariant probability measure:  $\nu_f$ .

*Proof.* First notice that  $\nu_f$  is contained in  $E(0)$  and is invariant.

Suppose  $\mu$  is an invariant probability measure contained in  $E(0)$ . By the definition of  $E(0)$ , the conditional measures of  $\mu$  along the partition  $\xi^u$  are  $\nu_{(\cdot)}^u$  almost everywhere. We claim that the disintegration of  $h_*(\mu)$  along the partition  $\xi^A$  equals to  $\omega_{(\cdot)}^u$ . The claim follows from the definition of  $\nu_{(\cdot)}^u$ : for every  $x \in \mathbb{T}^3$ ,

$$h_*(\nu_x^u) = \omega_{h(x)}^u.$$

Since among the invariant probability measures of  $A$ , only  $\omega$  admits such disintegration, the above claim implies that  $h_*(\mu) = \omega$ . Because  $(h^{-1})_*(\omega) = \nu_f$ , we complete the proof.  $\square$

#### 4.2. Mostly contracting center.

**Proposition 4.3.** *There is  $n_0 > 0$  and  $\alpha_0 > 0$  such that for any  $x \in \mathbb{T}^3$  and  $n \geq n_0$ ,*

$$(2) \quad \int_{\xi^u(x)} \log(df^n|_{E^c})(y) d\nu_x^u(y) \leq -\alpha_0 < 0.$$

*Proof.* We take  $\alpha_0 = -\lambda^c(\nu_f)/2 > 0$ . Suppose this proposition is false, then there is  $x_n \in \mathbb{T}^3$  and  $t_n \rightarrow \infty$  such that

$$\int_{\xi^u(x_n)} \log(df^{t_n}|_{E^c})(y) d\nu_{x_n}^u(y) \geq -\alpha_0.$$

But any limit of the sequence  $\frac{1}{t_n} \sum (f^i)_*(\nu_{x_n}^u)$  belongs to  $E(0)$  and is an invariant probability measure, hence coincides to  $\nu_f$ , this implies that

$$\frac{1}{t_n} \sum (f^i)_*(\nu_{x_n}^u) \rightarrow \nu_f.$$

Then

$$\begin{aligned} \frac{1}{t_n} \int_{\xi^u(x_n)} \log(df^{t_n}|_{E^c})(y) d\nu_{x_n}^u(y) &= \frac{1}{t_n} \int_{\xi^u(x_n)} \sum_{i=0}^{t_n-1} \log(df|_{E^c})(f^i(y)) d\nu_{x_n}^u(y) \\ &= \int_{\xi^u(x_n)} \log(df|_{E^c}) d\frac{1}{t_n} \sum_{i=0}^{t_n-1} (f^i)_*(\nu_{x_n}^u) \\ &\rightarrow \lambda^c(\nu_f), \end{aligned}$$

a contradiction.  $\square$

*Remark 4.4.* By the discussion in Subsection 2.4, for simplicity, we assume  $n_0 = 1$  from now on.

#### 4.3. Support of $\nu_f$ .

**Proposition 4.5.** *The support of  $\nu_f$  is  $u$  saturated, that is, it consists union of entire  $\mathcal{F}^u$  leaves. Moreover,  $\text{supp}(\nu_f)$  is a minimal  $\mathcal{F}^u$  foliation component.<sup>4</sup> And there are  $r_0 > 0$ ,  $0 < a_0, b_0 < 1$  and  $C > 0$  such that for every  $x \in \mathbb{T}^3$ , there is a set  $\Gamma_x \subset \xi^u(x)$  satisfying:*

$$(3) \quad \begin{aligned} &\text{(a) } \nu_x^u(\Gamma_x) > a_0; \\ &\text{(b) for every } y \in \Gamma_x, \\ &\|df^n|_{E^{cs}(y)}\| \leq Cb_0^n, \end{aligned}$$

and for any  $z \in \mathcal{F}_{r_0}^{cs}(y)$  and every  $n \geq 0$ ,  $d(f^n(y), f^n(z)) < Cb_0^n$ .

*Remark 4.6.* Replacing  $f$  by some power, we may assume that  $C = 1$ .

*Proof.* Because  $\text{supp}(\nu_f)$  is  $f$  invariant, to prove that  $\text{supp}(\nu_f)$  is  $\mathcal{F}^u$  saturated, it suffices to show that for any  $x \in \text{supp}(\nu_f)$ ,  $\xi^u(x) \subset \text{supp}(\nu_f)$ .

For the linear Anosov diffeomorphism  $A$ , it is well known that for every  $z \in \mathbb{T}^3$ ,  $\text{supp}(\omega_z^u) = \xi^A(z)$ , this is because, the maximal measure coincides with the Lebesgue measure. Recall that  $h$  is injective between unstable leaves, and for  $\nu_f$  typical point  $y$ ,  $h_*(\nu_y^u) = \omega_{h(y)}^u$ , we have that  $\text{supp}(\nu_y^u) = \xi^u(y)$ .

Take  $\nu^u$  typical points  $y_n$  converging to  $x$ , since  $\text{supp}(\nu_f)$  is a compact set, this implies that  $\xi^u(x) = \lim \xi^u(y_n) \subset \text{supp}(\nu_f)$ . Hence  $\text{supp}(\nu_f)$  is  $\mathcal{F}^u$  saturated.

Now we are ready to show that  $\text{supp}(\nu_f)$  is indeed a minimal  $\mathcal{F}^u$  component.

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<sup>4</sup> $\text{supp}(\nu_f)$  is a minimal  $\mathcal{F}^u$  foliation component was proven in [12] with a different argument.



Take  $x$  any  $\nu_f$  regular point. Then by Birkhoff ergodic theorem,

$$(4) \quad \frac{1}{n} \log |df^n|_{E^c(x)}| = \frac{1}{n} \sum_{i=0}^{n-1} \log |df|_{E^c(f^i(x))}| \rightarrow \lambda^c(\nu_f).$$

We claim that there is  $x' = f^j(x)$  for some  $j > 0$  such that

$$(5) \quad \frac{1}{n} \log |df^n|_{E^c(x')}| < e^{n\lambda^c(\nu_f)/2} \text{ for any } n \geq 0.$$

Suppose this claim is false, then there is  $n_0 > 0$  such that  $|df^{n_0}|_{E^c(x)}| > e^{n_0\lambda^c(\nu_f)/2}$ . And there is  $n_1 > 0$  such that  $|df^{n_1}|_{E^c(f^{n_0}(x))}| > e^{n_1\lambda^c(\nu_f)/2}$ . ... Then the sequence of positive integers  $n_j$  ( $j \geq 0$ ) satisfies

$$|df^{\sum_{j=0}^m n_j}|_{E^c(x)}| > e^{(\sum_{j=0}^m n_j)\lambda^c(\nu_f)/2},$$

a contradiction to (4).

We need the following context of Liao's shadowing lemma for  $f \in \mathcal{D}(A)$ :

**Lemma 4.7.** *Liao's shadowing Lemma*[17]

*There exists  $\varepsilon_0 > 0$ ,  $L > 0$ , such that for any periodic pseudo orbit*

$$\{x, f(x), \dots, f^{m-1}(x)\}$$

*with  $d(f^m(x), x) < \varepsilon_0$  and*

$$\|df^i|_{E^c(x)}\| < e^{i\lambda^c(\nu_f)/2} \text{ for any } 0 \leq i \leq n,$$

*Then there exists a periodic orbit  $p$  such that  $f^m(p) = p$ , and  $d(f^i(x), f^i(p)) \leq Ld(f^m(x), x)$  for every  $0 \leq i \leq m-1$ .*

Because  $x$  is a  $\nu_f$  regular point, there is  $m$  such that  $d(x', f^m(x'))$  can be arbitrarily small. Applying Liao's shadowing lemma, we obtain a periodic point  $p$  which is arbitrarily close to  $x'$ . Moreover, because  $d(f^i(x'), f^i(p)) \leq Ld(f^m(x'), x')$ , we can assume that

$$\frac{1}{n} \log |df^n|_{E^c(p)}| < e^{n\lambda^c(\nu_f)/4} \text{ for any } 0 \leq n \leq m-1.$$

Since  $p$  has period  $m$ , the above inequality holds for all  $n \in \mathbb{Z}$ . As a well-known fact (see for example [1][Lemma 2.7]),  $p$  has uniform size of stable manifold, that is, there is  $r > 0$  such that the stable manifold of  $p$  contains  $\mathcal{F}_r^{cs}(p)$ .

Thus we have  $W^s(p) \cap \mathcal{F}^u(x) \neq \emptyset$ . By the  $\lambda$ -lemma and  $\text{supp}(\nu_f)$  is  $\mathcal{F}^u$  saturated, which implies that  $p \in \text{supp}(\nu_f)$ . As a consequence,  $\text{Cl}(\mathcal{F}^u(\text{Orb}(p))) \subset \text{supp}(\nu_f)$ .

It is easy to see that  $\text{Cl}(\mathcal{F}^u(\text{Orb}(p)))$  admits an invariant probability measure in  $E(0)$ . In fact,  $\frac{1}{n} \sum_{i=0}^{n-1} \nu_p^u$  belongs to  $E(0)$  and is supported on  $\text{Cl}(\mathcal{F}^u(\text{Orb}(p)))$ , so does any of its limit. Because  $E(0)$  consists a unique invariant measure, which is the maximal measure  $\nu_f$ , it follows that  $\text{supp}(\nu_f) \subset \text{Cl}(\mathcal{F}^u(\text{Orb}(p)))$ .

In the following we claim that

$$\text{supp}(\nu_f) = \text{Cl}(\mathcal{F}^u(p)).$$

To prove this claim, observe that  $f^{\pi(p)}$  is still a derived Anosov diffeomorphism. Therefore it admits a unique maximal measure  $\nu_f$ . Since  $p \in \text{supp}(\nu_f)$  is a fixed point of  $f^{\pi(p)}$ , by similar argument as above we get  $\text{supp}(\nu_f) = \text{Cl}(\mathcal{F}^u(p))$ , as we claimed.

Let us continue the proof that  $\text{supp}(\nu_f)$  is a minimal  $\mathcal{F}^u$  component. For any  $x \in \text{supp}(\nu_f)$ , any limit of the sequence of probability measures

$$\frac{1}{n} \sum_{i=0}^{n-1} f_*^i \nu_x^u$$

is an invariant measure contained in  $E(0)$  and is supported on  $\text{Cl}(\bigcup_{i \in \mathbb{Z}} (\mathcal{F}^u(f^i(x))))$ . Because  $E(0)$  contained only one invariant probability measure which is  $\nu_f$ ,

$$\text{supp}(\nu_f) \subset \text{Cl}(\bigcup_{i \in \mathbb{N}} (\mathcal{F}^u(f^i(x)))).$$

But  $x \in \text{supp}(\nu_f)$  and  $\text{supp}(\nu_f)$  is  $\mathcal{F}^u$  saturated, we have indeed equality

$$\text{supp}(\nu_f) = \text{Cl}(\bigcup_{i \in \mathbb{N}} (\mathcal{F}^u(f^i(x)))).$$

Moreover, by the same proof, we can show that for any  $n > 0$ ,

$$(6) \quad \text{supp}(\nu_f) = \text{Cl}(\bigcup_{i \in \mathbb{N}} (\mathcal{F}^u(f^{in}(x)))).$$

We further claim that there is  $R > 0$  such that for each  $x \in \text{supp}(\nu_f)$ ,  $\mathcal{F}^u(x) \pitchfork W_R^s(p) \neq \emptyset$ . The proof is standard, by the compactness of  $\text{supp}(\nu_f)$  and the invariance of  $W^s(p)$ . More precisely, by (6), for any  $x$ , there is  $n_x \in \mathbb{N}$  such that  $f^{\pi(p)n_x}(\mathcal{F}^u(x)) \pitchfork W^s(p) \neq \emptyset$ . Take an iteration of  $f^{-\pi(p)n_x}$ , and choose  $R_x > 0$  sufficiently large, we have  $\mathcal{F}^u(x) \pitchfork W_{R_x}^s(p) \neq \emptyset$ . Furthermore, by continuation of foliation, this property holds for every point in a small neighborhood  $U_x$  of  $x$ . Take a finite cover of  $\text{supp}(\nu_f)$  by  $\{U_{x_i}; x_i \in \text{supp}(\nu_f)\}$ , and take  $R = \max\{R_{x_i}\}$ , we get the desired result.

As a corollary, for every  $x \in \text{supp}(\nu_f)$ ,  $p \in \text{Cl}(\mathcal{F}^u(x))$ . This is because for  $n \rightarrow \infty$ , there is  $a_n \in \mathcal{F}^u(f^{-n}(x)) \pitchfork W_{R_0}^s(p)$ . Then  $f^n(a_n) \in \mathcal{F}^u(x)$  and  $f^n(a_n) \rightarrow p$ . This implies that  $\text{supp}(\nu_f)$  is a  $\mathcal{F}^u$  minimal component.

Let  $b_1 = e^{\lambda^c(\nu_f)/2}$ . Denote by  $\Gamma$  the set of  $x$  such that for any  $n \geq 0$ ,

$$\|df^n|_{E^{cs}(x)}\| \leq b_1^n.$$

Then  $\Gamma$  is a compact set, and by (5), for every  $\nu_f$  regular point  $x$ , there is  $j > 0$  such that  $f^j(x) \in \Gamma$ . By Birkhoff ergodic theorem,  $\nu_f(\Gamma) > 0$ .

Choose any  $b_1 < b_0 < 1$ , by [1, Lemma 2.7], there is  $r_1 > 0$  such that every  $x \in \Gamma$  has uniform size of stable manifold, which contains  $\mathcal{F}_{2r_1}^{cs}$ , more precisely, for any  $y, z \in \mathcal{F}_{2r_1}^{cs}(x)$  and  $n \geq 0$ ,

$$d(f^n(y), f^n(z)) < b_0^n.$$

Moreover, we may assume the bundles  $E^s, E^c, E^u$  in the partial hyperbolicity splitting are orthogonal. Then by the continuation of  $\|df|_{E^c(x)}\|$ ,

$$\|df^n|_{E^{cs}(y)}\| = \|df^n|_{E^c(y)}\| \leq b_0^n.$$

Take a  $\nu_f$  generic point  $x_0$  such that  $\nu_{x_0}^u(\Gamma) = a_1 > 0$ . There is a neighborhood  $V$  of  $\xi^u(x_0)$ , such that for any point  $y \in V$ , the holonomy map  $\mathcal{H}_{x_0, y}^{cs}$  between  $\xi^u(x_0)$  and  $\xi^u(y)$  satisfies  $d(z, \mathcal{H}_{x_0, y}^{cs}(z)) < r_1$ . Because  $\xi^u$  is invariant under center-stable holonomy map, for any  $y \in V$ , denote by  $\Gamma_y = \mathcal{H}_{x_0, y}^{cs}(\Gamma \cap \xi^u(x_0))$ , then  $\nu_y^u(\Gamma_y) = a_1$ .

By the minimality of the unstable foliation  $\mathcal{F}^u$  inside  $\text{supp}(\nu_f)$ , there is  $N > 0$ , such that for any  $z \in \text{supp}(\nu_f)$ ,  $f^N(\xi^u(z)) \cap V \neq \emptyset$ . Let  $C = (\max\|Df\|)^N$ , and  $r_0 = \frac{r_1}{C}$ . Then suppose  $z' \in f^N(\xi^u(z)) \cap V$ , let  $\Gamma_z = f^{-N}(\Gamma_{z'})$ . It is easy to see that  $\Gamma_z$  satisfies (b). It remains to prove item (a).

Note that  $\nu_z^u(\Gamma_z) = \nu_z^u(f^{-N}(\xi^u(z')))\nu_{z'}^u(\Gamma_{z'}) \geq \nu_z^u(f^{-N}(\xi^u(z'))a_1$ .

It is a well known fact from finite shift of symbolic dynamical system that,  $\nu_z^u(f^{-N}(\xi^u(z')))$  is uniformly bounded from 0, hence we complete the proof of this proposition.  $\square$

From now on, we fix  $x_0, \Gamma, 0 < b_1 < b_0 < 1$  and the neighborhood  $V$  as in Proposition 4.5.

## 5. LARGE DEVIATIONS

In this section we prove Theorem B in a more general form:

**Proposition 5.1.** *For every  $\phi \in C^0(M)$  with  $\nu_f(\phi) = 0$  and every  $\epsilon > 0$  there exists constants  $C_\epsilon, c_\epsilon > 0$  such that for every  $l \in E(R)$ ,*

$$l(|S_n(\phi)| > \epsilon n) \leq C_\epsilon e^{-c_\epsilon n}.$$

*Remark 5.2.* In the following, we will prove the above proposition for  $f^n$  when  $n$  is sufficiently large.

The proof of the above proposition consists of several lemmas. The main idea comes from [13].

**Lemma 5.3.** *For any continuous function  $A$  with  $\nu_f(A) < -\alpha_0 < 0$ , there is  $C_1 > 0$ , such that for any  $n > 0$  and  $x \in \mathbb{T}^3$ , we have*

$$\int_{\xi^u(x)} S_n(A) d\nu_x^u \leq -n\alpha_0/2 + C_1.$$

*Proof.* By an argument similar to Proposition 4.3 with  $\log(df^n|_{E^c})$  replaced by  $S_n(A)$ , there is  $n_0 > 0$  such that  $\int_{\xi^u(x)} S_{n_0}(A) d\nu_x^u \leq -n_0\alpha_0/2$ . In the following we claim that this lemma is true for  $n = kn_0$ . Then by taking  $C_1 = k|A|$ , We conclude the proof.

To prove this claim, we use induction. Suppose this lemma holds for  $n = n_0, \dots, (k-1)n_0$ ,

$$\begin{aligned} \int_{\xi^u(x)} S_{kn_0}(A) d\nu_x^u &= \\ \int_{\xi^u(x)} S_{n_0}(A) d\nu_x + \int_{f^{n_0}(\xi^u(x))} S_{(k-1)n_0}(A) d(f_*^{n_0} \nu_x^u). \end{aligned}$$

Let  $f^{n_0}(\xi^u(x)) = \bigcup \xi^u(x_j)$ . Then the second term equals

$$\sum_j c_j \int_{\xi^u(x_j)} S_{(k-1)n_0}(A) d\nu_{x_j}^u,$$

where  $c_j = \nu_x^u(f^{-n_0}(\xi^u(x_j)))$ . By induction,

$$\int_{\xi^u(x_j)} S_{(k-1)n_0}(A) d\nu_{x_j}^u \leq -(k-1)n_0\alpha_0/2.$$

Summing over  $j$ , we complete the proof.  $\square$

By the uniform contraction of  $f^{-n}$  restrict to  $\mathcal{F}^u$  and continuity of  $A$  we have:

**Proposition 5.4.** *There is  $C > 0$  such that for any  $\varepsilon > 0$ , there exists an  $n_\varepsilon > 0$  such that for any  $x \in \mathbb{T}^3$ ,  $n \geq n_\varepsilon$  and any  $y_1, y_2 \in f^{-n}(\xi^u(x))$ ,*

$$|S_n(A)(y_1) - S_n(A)(y_2)| \leq (n + C)\varepsilon.$$

As a corollary of Lemma 5.3 and Proposition 5.4:

**Corollary 5.5.** *There is  $\alpha_1 > 0$  and  $n_1 \in \mathbb{N}$  such that for every  $n > n_1$  and for any  $x \in \mathbb{T}^3$ , write  $f^n(\xi^u(x)) = \bigcup_j \xi^u(x_j)$ , we have,*

$$\sum_j c_j \max_{f^{-n}(\xi^u(x_j))} S_n(A) \leq -\alpha_1 < 0,$$

where  $c_j = \nu_x^u(f^{-n}(\xi^u(x_j)))$ .

By the discussion in Subsection 2.4, replace  $f$  by its iteration  $f^n$  for  $n$  large, we think the above corollary works for any  $n \geq 1$ . With this assumption, we have that:

**Lemma 5.6.** *If  $s$  is small enough, there is a constant  $\theta_1 < 1$  such that for every  $n \geq 1$ ,*

$$\sum_j c_j \exp \left( s \max_{f^{-n}(\xi^u(x_j))} S_n(A) \right) \leq \theta_1.$$

*Proof.* Consider the function  $r_x(s) = \sum_j c_j \exp(s \cdot \max_{f^{-n}(\xi^u(x_j))} S_n(A))$ . Then  $r_x(0) = 1$ ,  $\frac{dr_x}{ds}(0) \leq -\alpha_1 < 0$ , and  $\left| \frac{d^2 r_x}{ds^2}(s) \right|$  is uniformly bounded for any  $x \in \mathbb{T}^3$  and  $s \in [0, 1]$ . The last inequality comes from the fact that the items inside the sum is uniformly bounded.

Then the lemma follows immediately from the above observation.  $\square$

**Corollary 5.7.** *For any  $n > 0$ , and any  $x \in \mathbb{T}^3$ , denote by  $f^n(\xi^u(x)) = \cup_j \xi^u(x_j)$  and  $c_j = \nu_x^u(f^{-n}(\xi^u(x_j)))$ , then*

$$(7) \quad \sum_j c_j \exp \left( s \max_{f^{-n}(\xi^u(x_j))} S_n(A) \right) \leq \theta_1^n.$$

*Proof.* The proof comes from an induction. By the previous lemma, (7) is valid for  $k = 1$ . Now assume that it is correct for all  $k \leq n - 1$ . Let  $f(\xi^u(x)) = \cup_i \xi^u(y_i)$ ,  $f^{n-1}(\xi^u(y_i)) = \cup_j \xi^u(x_{ij})$ ,  $b_i = \nu_x^u(f^{-1}(\xi^u(y_i)))$  and  $c_{ij} = \nu_x^u(f^{-n}(\xi^u(x_{ij})))$ . Then  $f^n(\xi^u(x)) = \cup_{ij} \xi^u(x_{ij})$ , and

$$\begin{aligned} \sum_{ij} c_{ij} \exp \left( s \max_{f^{-n}(\xi^u(x_{ij}))} S_n(A) \right) &= \\ \sum_{ij} b_i c_{ij} \exp \left( s \max_{f^{-n}(\xi^u(x_{ij}))} S_n(A) \right) &\leq \\ \sum_i b_i \exp \left( s \max_{f^{-1}(\xi^u(y_i))} S_1(A) \right) \sum_j c_{ij} \exp \left( s \max_{f^{-(n-1)}(\xi^u(x_{ij}))} S_{n-1}(A) \right) &\leq \\ \sum_i b_i \exp \left( s \max_{f^{-1}(\xi^u(y_i))} S_1(A) \right) \theta_2^{n-1} &\leq \\ \theta_1^n. \end{aligned}$$

$\square$

In particular if we take  $A = \log |df|_{E^c}$  we get:

**Theorem 5.8.** *There exist  $s > 0$  and  $0 < \theta_1 < 1$  such that for any  $x \in \mathbb{T}^3$  and any  $n > 0$ ,*

$$\int_{\xi^u(x)} (df^n|_{E^c})^s d\nu_x^u \leq \theta_1^n.$$

*Proof of Proposition 5.1.* First we verify the proposition for  $l \in E_1(0)$ .

Given function  $\phi$  with  $\nu_f(\phi) = 0$ , for  $\epsilon > 0$  define  $\tilde{\phi}_\epsilon = \phi - \epsilon$ . Then we can apply Corollary 5.7 on  $\tilde{\phi}_\epsilon$  and get for some  $0 < \theta_\epsilon < 1$ ,

$$\sum_j c_j \exp \left( s \max_{f^{-n}(\xi^u(x_j))} S_n(\tilde{\phi}_\epsilon) \right) \leq \theta_\epsilon^n.$$

This implies that

$$l(\exp(s(S_n(\phi) - n\epsilon))) \leq \theta_\epsilon^n$$

for every  $l \in E(0)$ . Now we apply Chebyshev's inequality and obtain

$$l(S_n(\phi) \geq n\epsilon) \leq \theta_\epsilon^n.$$

The same argument applying to the lower bound of  $S_n(\phi)$  gives

$$l(|S_n(\phi)| \geq n\epsilon) \leq \theta_\epsilon''^n,$$

this finishes the proof for  $l \in E(0)$ . Now, given any  $l \in E_1(R)$  we write  $n = (1 - \delta)n + \delta n$  for some  $\delta > 0$  small. we have

$$l(|S_n(\phi)| \geq n\epsilon) \leq l(|S_{\delta n}(\phi)| \geq \frac{n\epsilon}{2}) + l(|S_{(1-\delta)n}(\phi) \circ f^{\delta n}| \geq \frac{n\epsilon}{2}).$$

The first term is 0 if  $\delta$  is chosen small enough. To deal with the second term we assume that

$$l(A) = \int_{\xi^u(x)} A(z) e^{G(z)} d\nu_x^u(z).$$

Write  $f^{\delta n}\xi(x) = \bigcup_i \xi(x_i)$  and denote by  $l_i$  measures on  $\xi(x_i)$  with

$$l_i(A) = \int_{\xi^u(x_i)} A(z) e^{G(f^{-\delta n}z)} d\nu_x^u(z).$$

Choose some  $z_i \in \xi^u(x_i)$  we get that

$$\begin{aligned} |l_i(A) - \nu_{x_i}^u(A)| &= \int_{\xi^u(x_i)} A(z) (e^{G(f^{-\delta n}z)} - 1) d\nu_x^u(z) \\ &= (e^{G(f^{-\delta n}z_i)} - 1) \int_{\xi^u(x_i)} A(z) d\nu_x^u(z) \\ &\leq C\tilde{\theta}^{\delta n} \end{aligned}$$

where  $C$  and  $\tilde{\theta}$  depend on  $G$  and  $A$  but not on  $i$ . As a result

$$\begin{aligned} &l\left(|S_{(1-\delta)n}(\phi) \circ f^{\delta n}| \geq \frac{n\epsilon}{2}\right) \\ &= \sum_i c_i l_i\left(|S_{(1-\delta)n}(\phi)| \geq \frac{n\epsilon}{2}\right) \\ &\leq \sum_i c_i \nu_{x_i}^u\left(|S_{(1-\delta)n}(\phi)| \geq \frac{n\epsilon}{2}\right) + C\tilde{\theta}^{\delta n} \\ &\leq C\tilde{\theta}^n. \end{aligned}$$

□

## 6. COUPLING ARGUMENT

The rest of the proof in this paper is quite similar to the argument of [12][Section 6]. See also [38].

We want to show that for large  $n$  and any  $l_1, l_2 \in E(R)$ ,  $\mathcal{T}^n(l_1)$  is close to  $\mathcal{T}^n(l_2)$ . First we consider the case when  $l_1$  and  $l_2$  both belong to  $E_1(0)$ , that is,  $l_i = \nu_{x_i}^u$ . Denote by  $Y_i = \xi^u(x_i) \times I$  where  $I = [0, 1]$ . Equip  $Y_i$  with the measure  $dm_i = d\nu_{x_i}^u \times dt$ .

**Lemma 6.1.** *There is a measure preserving map  $\tau : Y_1 \rightarrow Y_2$ , a function  $R : Y_1 \rightarrow \mathbb{N}$  and constants  $C_1, C_2 > 0$ ,  $\rho_1 < 1$ ,  $\rho_2 < 1$  such that*

- (A) *If  $\tau(y_1, t_1) = (y_2, t_2)$ , then for  $n \geq R(x_1, t_1)$ ,*
- (8)  $d(f^n(y_1), f^n(y_2)) \leq C_1 \rho_1^{n-R}.$
- (B)  $m_1(R > N) \leq C_2 \rho_2^N.$

The proof of Lemma 6.1 occupies Section 7.

Let  $\|l\|_\gamma$  denote the norm of  $l$  on the space  $C^\gamma(X)$

**Corollary 6.2.** *There exist  $C_3 > 0$ ,  $\rho_3 < 1$  such that for any  $n > 0$ , and any  $l_1, l_2 \in E(0)$ ,  $\|\mathcal{T}^n(l_1 - l_2)\|_\gamma \leq C_3 \rho_3^n.$*

*Proof.* It suffices to prove for every  $l_i \in E_1(0)$ . We have

$$(\mathcal{T}^n l_j)(A) = \int_{Y_j} A(f^n(y_j)) dm_j(y_j, t_j).$$

Let  $(y_2, t_2) = \tau(y_1, t_1)$ . Then

$$\mathcal{T}^n(l_1 - l_2)(A) \leq \int_{Y_1} |A(f^n(y_1)) - A(f^n(y_2))| dm_1(y_1, t_1).$$

Let  $Z(n) = \{z : R(z) \leq \frac{n}{2}\}$  then

$$\begin{aligned} & |\mathcal{T}^n(l_1 - l_2)(A)| \\ (9) \quad & \leq \int_{Z(n)} |A(f^n(y_1)) - A(f^n(y_2))| dm_1(y_1, t_1) + 2\|A\|_0 m_1(Y_1 \setminus Z(n)) \\ & \leq \|A\|_\gamma ((C_1 \rho_1^{\frac{n}{2}})^\gamma + 2C_2 \rho_2^{\frac{n}{2}}). \end{aligned}$$

□

Replace  $l_2$  by  $\nu_f$ , we have that

**Corollary 6.3.** *For any  $l \in E(0)$  and any  $A \in C^\gamma(M)$ ,  $n > 0$ ,*

$$\left| \int A(f^n(x)) dl(x) - \nu(A) \right| \leq C_3 \rho_3^n \|A\|_\gamma.$$

### 6.1. Proof of the main results.

*Proof of Theorem A.* Consider  $l \in E(R)$ . By Proposition 4.1, there exists  $\tilde{l} \in E(0)$  such that

$$\|\mathcal{T}^{\frac{n}{2}} l - \tilde{l}\| \leq \text{Const} \cdot e^{-\frac{\lambda_5 n}{2}}.$$

Hence, there is  $0 < \tau = \max\{e^{-\frac{\lambda_5}{2}}, \rho_3\} < 1$  such that

$$\|\mathcal{T}^n l - \nu\| \leq \text{Const} \cdot e^{-\frac{\lambda_5 n}{2}} + \|\mathcal{T}^{\frac{n}{2}} \tilde{l} - \nu_f\| \leq C_4 \tau^n.$$

To finish the proof of Theorem A, one only need take  $l = \phi \cdot \nu_f$ .

□

## 7. COUPLING ALGORITHM

In the beginning, we fix

- $x_0$  be the point chosen in the proof of Proposition 4.5 and  $V$  the small neighborhood of  $\xi^u(x_0)$  which satisfies the condition (12) below;
- $b_0$  in Proposition 4.5;
- $s, \theta_1$  in Theorem 5.8.

Let  $\lambda$  be close to 0 such that  $e^{-\lambda s} > \theta_1$  and

$$(10) \quad e^{-\lambda} > b_0$$

By Corollary 5.7, if  $K > 0$  is large enough, then

$$(11) \quad q_1 = \max_x \nu_x^u(U(\xi^u(x))) < 1,$$

where  $U(\xi^u(x)) = \{y \in \xi^u(x) : \exists n > 0, z \in (f^{-n} \xi^u)(y) \text{ such that } (df^n|_{E^c})(z) \geq K e^{-\lambda n}\}$ .

As explained before, we assume the bundles  $E^s, E^c, E^u$  in the partial hyperbolic splitting are orthogonal. Then for any  $j > 0$  and  $x \in \mathbb{T}^3$ ,

$$\|(df^j|_{E^{cs}})(x)\| = |(df^j|_{E^c})(x)|.$$

There is  $\delta > 0$  such that if  $d(x, y) < \delta$ , then  $\|(df|_{E^{cs}})(y)\| \leq e^{\lambda/2} \|(df|_{E^{cs}})(x)\|$ . Let  $\varepsilon \leq \frac{\delta}{2K}$ . We may shrink the neighborhood  $V$  such that for any  $y_1, y_2 \in V$ , and  $z \in \xi^u(y_1)$ ,

$$(12) \quad d(z, \mathcal{H}_{y_1, y_2}^{cs}) < \varepsilon.$$

**7.1. First run.** In the first run, we define the map between  $P_j^\infty$  of  $Y_j$ . For the points where  $\tau$  is not defined, we define a stopping time  $s(y)$  such that the set  $P_j^n = \{y \in Y_j : s(y) = n\}$  is of the form  $f^{-n}(\bigcup_k Y_{jnk})$ , and  $m_1(P_1^n) = m_2(P_2^n)$ .

Because  $\text{supp}(\nu_f)$  is a  $\mathcal{F}^u$  minimal component (Proposition 4.5), there is  $n_0$  such that

- $f^{n_0}(\xi^u(x_1))$  and  $f^{n_0}(\xi^u(x_2))$  both cross  $V$ ;
- there exist  $x_{j,1} \in f^{n_0}(\xi^u(x_j)) \cap V$  for  $j = 1, 2$  and both points belong to a 'Markov' component  $M_{i_0}$ .

Then

$$d_{cs}(y, \mathcal{H}_{x_1, x_2}^{cs}(y)) \leq \varepsilon,$$

where  $y \in \xi^u(x_1)$  and  $d_{cs}$  denotes the distance inside each  $cs$  leaf. Let  $\hat{c}_j = \nu_{x_j}^u(f^{-n_0}(\xi^u(x_{j,1})))$ ,  $(\bar{t}_1, \bar{t}_2) = (\frac{\hat{c}_2}{\hat{c}_1}, 1)$  if  $\hat{c}_2 \leq \hat{c}_1$ , and  $(\bar{t}_1, \bar{t}_2) = (1, \frac{\hat{c}_1}{\hat{c}_2})$  if  $\hat{c}_1 \leq \hat{c}_2$ . Define  $\bar{Y}_j = \xi^u(x_{j,1}) \times [0, \bar{t}_j]$ . Let  $s(y) = n_0$  for points of  $Y_j \setminus f^{-n_0}\bar{Y}_j$ .

We now proceed to define  $P_j^n$  inductively for  $n > n_0$ . Let  $Q_j^{n-1} = Y_j \setminus \bigcup_{m=n_0}^{n-1} P_j^m$ . We assume by induction that  $f^{n-1}Q_j^{n-1} = \bigcup_k Z_{jk(n-1)}$  where

$$Z_{jk(n-1)} = \xi^u(x_{jk(n-1)}) \times [0, \bar{t}_j],$$

and

$$(13) \quad m_1(f^{-(n-1)}Z_{1k(n-1)}) = m_2(f^{-(n-1)}Z_{2k(n-1)}),$$

$\xi^u(x_{1k(n-1)})$  and  $\xi^u(x_{2k(n-1)})$  belong to the same 'Markov' element and

$$(14) \quad d(x, \mathcal{H}^{cs}(x)) \leq r_{n-1} \text{ where } r_n = Ke^{-\lambda n/2}.$$

As a well-known property of the Markov partition, we can write  $f(\xi^u(x_{jk(n-1)})) = \bigcup_l \xi^u(x_{jlk n})$  for  $j = 1, 2$ , such that  $\xi^u(x_{1lk n})$  and  $\xi^u(x_{2lk n})$  belong to the same Markov component for every  $l$ . Denote by  $\beta_{lkn} = \|df|_{E^c}|_{f^{-n}(\xi^u(x_{1lk n}))}\|$ . If  $\beta_{lkn} > Ke^{-\lambda n}$  let  $s(y) = n$  on  $f^{-n}(\xi_{1lk n}^u) \times [0, \bar{t}_1]$ . Otherwise let  $Z_{jlk n} = \xi^u(x_{jlk n}) \times [0, \bar{t}_j]$ .

**Lemma 7.1.**

$$(15) \quad m(f^{-n}(Z_{1jkn})) = m(f^{-n}(Z_{2jkn})).$$

*Proof.* Because  $\frac{\bar{t}_1}{\bar{t}_2} = \frac{\hat{c}_2}{\hat{c}_1} = \frac{\nu_{x_2}^u(f^{-1}(\xi^u(x_{21})))}{\nu_{x_1}^u(f^{-1}(\xi^u(x_{11})))}$ , this follows from the fact that

$$\frac{\nu_{x_1}^u(f^{-n}(\xi^u(x_{1lk n})))}{\nu_{x_2}^u(f^{-n}(\xi^u(x_{2lk n})))} = \frac{\nu_{x_1}^u(f^{-1}(\xi^u(x_{11})))}{\nu_{x_2}^u(f^{-1}(\xi^u(x_{21})))}$$

Observe that for  $j = 1, 2$ ,

$$\frac{\nu_{x_j}^u(f^{-n}(\xi^u(x_{jlk n})))}{\nu_{x_j}^u(f^{-1}(\xi^u(x_{j1})))} = \nu_{x_{j1}}^u(f^{-(n-1)}(\xi^u(x_{jlk n}))).$$

Then this lemma is a corollary of the fact that  $(\mathcal{H}_{x_{11}, x_{21}}^{cs})_*(\nu_{x_{11}}^u) = \nu_{x_{21}}^u$  (Proposition 2.6), and  $\mathcal{H}_{x_{11}, x_{21}}^{cs}(f^{-(n-1)}(\xi^u(x_{1lk n}))) = f^{-(n-1)}(\xi^u(x_{2lk n}))$ .  $\square$

To complete the first run, we still need to show that  $d(x, \mathcal{H}^{cs}(x)) \leq r_n$  for  $(x, t) \in Z_{1jkn}$ , which is a corollary of the following lemma

**Lemma 7.2.** [12, Lemma 8.1] *If  $x \in M$  and  $n > 0$  are such that for any  $0 \leq j < n$ ,  $(df^j|_{E^c})(x) \leq Ke^{-\lambda j}$  then for any  $0 \leq j \leq n$ ,*

$$f^j(\mathcal{F}_{\varepsilon}^{cs}(x_0)) \subset \mathcal{F}_{r_j}^{cs}(f^j(x_0)),$$

where  $r_j = Ke^{-\lambda n/2}$ .

Thus in each run,  $\tau$  is not defined on  $(x, t) \in Y_1$  for  $s(x) = n$  by three different reasons:

- (a)  $n = n_0$  and  $f^{n_0}(x) \notin \xi^u(x_{11})$ ;
- (b)  $n = n_0$  and  $f^{n_0}(x) \in \xi_{x_{11}}^u$ , but  $t > \bar{t}_1$ ;
- (c) In the step  $n > n_0$ ,  $\|df|_{E^c}|_{f^{-n}(\xi^u(x))}\| > Ke^{-\lambda n}$ .

We only cut the height  $I$  to  $\bar{t}_1$  in the step  $n = n_0$ . Thus  $P_1^\infty = Y_1 \setminus \bigcup_n P_1^n$  is a union of vertical intervals of the form  $(x, [0, \bar{t}_1])$ . We define  $\tau : P_1^\infty \rightarrow P_2^\infty$  such that for any  $(x, t) \in P_1^\infty$ , denote by  $y = f^{-1} \circ \mathcal{H}_{x_{11}, x_{21}}^{cs} \circ f(x)$ , then

$$(16) \quad \tau((x, t)) = (y, \frac{\hat{c}_1}{\hat{c}_2}t) \text{ and } R(x, t) = n_0.$$

**Lemma 7.3.** *There is  $a > 0$  does not depend on  $x_1, x_2$  such that  $m_1(P_1^\infty) > a$ .*

*Proof.* We claim that  $f^{-n_0}(\Gamma_{x_{11}}) \times [0, \bar{t}_1] \subset P_1^\infty$ .

Because  $\Gamma_{x_{11}} \subset \xi^u(x_{11})$ , it does not fit the situation of case (a) above. And by (b) of Proposition 4.5 ((3)), the points in  $\Gamma_{x_{11}}$  also do not fit the situation of case (c), thus we finish the proof of the claim.

By (a) of Proposition 4.5, there is  $a_0 > 0$  such that  $\nu_{x_1}^u(\Gamma_{x_{11}}) > a_0$ . Moreover, there are only finitely many values of  $\hat{c}_j$  ( $j = 1, 2$ ), so do  $\bar{t}_j$ , take the minimal value denote by  $t_0$ . Let  $a = a_0 \cdot t_0$ , the proof is complete.  $\square$

**Lemma 7.4.**  $\tau|_{P_1^\infty}$  is measure preserving.

*Proof.* One only need to show that the map

$$f^{-1} \circ \mathcal{H}_{x_{11}, x_{21}}^{cs} \circ f : f^{-1}(\xi^u(x_{11})) \rightarrow f^{-1}(\xi^u(x_{21}))$$

has Jacobian

$$\frac{\hat{c}_2}{\hat{c}_1} = \frac{\nu_{x_2}^u(f^{-1}(\xi^u(x_{21})))}{\nu_{x_2}^u(f^{-1}(\xi^u(x_{11})))}.$$

This fact comes easily from the fact that  $\xi^u$  is an increasing Markov partition and because  $\nu^u$  is preserved by the center-stable holonomy.  $\square$

**7.2. Algorithm.** Our algorithm will work recursively.

Note that  $P_j^n = \{y \in Y_j : s(y) = n\}$  is the form of  $f^{-n}(\bigcup_k Y_{jnk})$ , and  $m_1(P_1^n) = m_2(P_2^n)$ . Then we can use our algorithm again to couple  $P_1^n$  to  $P_2^n$ . We first chop each  $Y_{jnk}$  into several pieces so that the resulting collections  $\tilde{Y}_{jnl}$  satisfies  $\bigcup_k Y_{jnk} = \bigcup_l \tilde{Y}_{jnl}$  and  $m_1(\tilde{Y}_{1nl}) = m_2(\tilde{Y}_{1nl})$ . Let  $f^{-n}\tilde{Y}_{jnl} = U_{jnl} \times I_{jnl}$ . Denote  $c_{jnl} = \nu_{x_j}^u(U_{jnl})$ . Let  $\Delta_{jnl}$  be the map  $\Delta_{jnl}(x, t) = (f^n(x), r_{jnl}(t))$  where  $r_{jnl}$  is the affine isomorphism between  $I_{jnl}$  and  $[0, c_{jnl}|I_{jnl}|]$ . We now call our algorithm recursively to produce maps  $\tau_{nl} : \Delta(f^{-n}\tilde{Y}_{1nl}) \rightarrow \Delta(f^{-n}Y_{2nl})$  and  $R_{nl} : \Delta(f^{-n}\tilde{Y}_{1nl}) \rightarrow \mathbb{N}$  satisfying the conditions of Lemma 6.1. We set

$$\tau(x, t) = \begin{cases} \tau_{\text{first run}}(x, t), & \text{if } (x, t) \in P_1^\infty; \\ \Delta_{2nl}^{-1} \circ \tau_{nl} \circ \Delta_{1nl}, & \text{if } (x, t) \in f^{-n}\tilde{Y}_{1nl}. \end{cases}$$

$$R(x, t) = \begin{cases} R_{\text{first run}}(x, t), & \text{if } (x, t) \in P_1^\infty; \\ n + R_{nl}(\Delta_{1nl}(x, t)) & \text{if } (x, t) \in f^{-n}\tilde{Y}_{1nl}. \end{cases}$$

Let us now describe the first run of our algorithm. We need to verify four things:



- (I)  $\tau$  is defined on a set of whole measure in  $Y_1$ ;
- (II)  $\tau$  is measure preserving;
- (III)  $\tau$  satisfies (A) of Lemma 6.1;
- (IV)  $\tau$  satisfies (B) of Lemma 6.1.

As a consequence of Lemma 7.3,  $\tau$  is defined on a set of whole measure in  $Y_1$ . (II) is a consequence of Lemma 7.4. Moreover, (III) comes directly from the construction (see (14)), if we take  $\rho_1 = K\varepsilon e^{-\lambda/2}$ .

**7.3. Coupling time.** Here we prove (B) of Lemma 6.1. As a summary of the above discussion, we have:

**Lemma 7.5.** *There are constants  $q, C_0 > 0, \rho_0 < 1$  such that for any pair  $Y_1, Y_2$*

- (H1)  $m_1(P_1^\infty) \geq a$ ;
- (H2)  $m_1(P_1^n) \leq C_0 \rho_0^n$ .

*Proof.* (H1) is exactly Lemma 7.3.

We begin by (H2), because for  $n > n_0$ , the only reason for  $(y, t)$  belongs to  $P_1^n$  is that  $\|df|_{E^c}|_{f^{-n}(\xi^u(y))}\| > Ke^{-\lambda n}$ . So by the choice of  $\lambda$ , the measure of such points is exponentially small by Corollary 5.7. □

Now represent  $R(y) = \sum_{j=1}^{k(y)} s_j(y)$ , where  $s_j(x)$  is the stopping time of the  $j$ th run of our algorithm. And  $T_k$  be the set where  $\tau$  is not defined after  $k$  runs of our algorithm and  $U_k = T_{k-1} \setminus T_k$ . Then  $m_1(R = n) = \sum_{i \leq [\delta n]} m_1(U_i) + \sum_{i > [\delta n]} m_1(U_i) = I + II$  for some  $\delta$  small

Then  $II < (1 - a)^{\delta n}$ . To estimate the first term, we only need consider

$$\begin{aligned}
 m_1(\{R = n\} \cap U_i) &= \sum_{(k_1, \dots, k_i): \sum k_j = n} m_1(\{s_i = k_i\}) \\
 (17) \quad &\leq \sum_{(k_1, \dots, k_i): \sum k_j = n} \left( \prod_{j=1}^i C_0 \rho_0^{k_j} \right) \\
 &\leq \binom{n}{i} C_0^i \rho_0^n.
 \end{aligned}$$

Because  $\binom{n}{[n\delta]} \approx e^{\varepsilon n}$  for some  $\varepsilon = \varepsilon(\delta)$  where  $\varepsilon(\delta) \rightarrow 0$  when  $\delta \rightarrow 0$ . Choosing  $\delta$  small enough such that

$$e^{\varepsilon(\delta)} C_0^\delta \rho_0 = \rho' < 1.$$

Then the first item is  $\leq [n\delta] \rho'^n$ .

We complete the proof of (B) in Lemma 6.1.

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